Estimation of Time-Varying Autoregressive Symmetric Alpha Stable Processes by Particle Filters

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***Abstract*—In this work, we propose a novel method to model *time-varying* autoregressive impulsive signals, which possess Symmetric Alpha Stable distributions. The proposed method is composed of a particle filter, which is capable of estimating the unknown, *time-varying* autoregressive coefficients and a Hybrid Monte Carlo method that is used for estimating the unknown statistical parameters of the Symmetric Alpha Stable Process. The performance of the proposed method is tested for different parameter values where the time variation of the autoregressive coefficients is taken to be as sinusoidal or random jumps. The successful performance of the developed method serves as a promising contribution in the modeling of impulsive signals, which are frequently seen in many areas, such as teletraffic in computer communications, radar and sonar applications and mobile communications.**

Index Terms— *α-stable* distributions, Bayesian estimation techniques, Markov Chain Monte Carlo, Particle Filtering

I. Introduction

THE necessity to model the impulsive data has led to an increasing attention in the signal processing community since the beginning of the last two decades. Until recently, most of the methods developed in the literature were based on the assumption that the data can be modeled by Gaussian distributions. However, in many application areas, such as the radar and sonar communications, financial time series modeling, telecommunications and teletraffic modeling, the distribution of the data does not fit to a Gaussian distribution [1]. Moreover, trying to model an impulsive data with a Gaussian distribution can lead to severe disadvantages, since the effects of the outliers are ignored in such a modeling. Additionally, these outliers cause the most extreme effects, when the physical nature of the data is considered, such as the cases in seismic, climatologic and oceanographic data. A remarkable example for this discussion can be the modeling of oceanic wave heights, where these are generally modeled by heavy-tailed distributions [2]. If one used a Gaussian distribution, instead; neglecting the outlier data could cause some catastrophic consequences during the process of meteorological predictions, such as the case that was witnessed in Holland in 1953, where an increase of 3.5m in the sea level led to the death of approximately 2000 people. Such a rise in water level is a remarkable example for an outlier and unfortunately, considering that as an event with low probability caused devastating outcomes.

In order to involve the information coming from the outliers, heavy-tailed modeling is widely used in literature. In general, Alpha Stable (*α-stable*) processes are used to model heavy-tailed distributions [3, 4, 5].

Another important issue in the modeling of heavy-tailed data is expressing its temporal dependency. Similar to the Gaussian cases, linear and nonlinear parametric temporal modeling of the impulsive data can be expressed in terms of Autoregressive (AR), Moving-Average (MA), Autoregressive Moving Average (ARMA) [3, 4, 6] or nonlinear AR models [7]. Among these, AR structure is widely used in teletraffic data modeling in computer communications [8].

Moreover, the time evolution of these data model parameters is also an active research area. According to the observations, there is an ongoing debate about the time dependency of the teletraffic data, where it is stated that the AR coefficients may be either time-varying or the process is self similar [9]. In another application area, the aforementioned parametric models can be used to model communication channels and it is well known that the characteristics, therefore the parametric model of these channels can vary over time, especially in wireless communications [10].

In literature, although there are different approaches to estimate the parameters of a *time-invariant* AR *α-stable* process, such as the Iteratively Reweighted Least Squares [7], Generalized Yule-Walker [3] and Markov Chain Monte Carlo (MCMC) based methods [11], a very limited number of works have been done to model *time-varying* AR (TVAR) *α-stable* process. In [12],

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Thavaneswaran and Peiris propose a penalized minimum dispersion method in case of a presumably *known* shape parameter (α) of the *α-stable* process which is taken to be larger than one, i.e. $\alpha > 1$.

In the case of a TVAR process with a non-Gaussian driving process, such as a Mixture of Gaussians or Laplacian with *known* statistical parameters, it is observed that the estimation of the unknown TVAR coefficients can be done successfully using Particle Filters [13, 14]. Apart from the signal modeling scheme, which is the case discussed so far, particle filters can also perform very well in the signal enhancement, where a *Gaussian* signal is embedded in a *Symmetric-α-stable (SαS)* noise process [15]. In [15] the underlying *Gaussian* process has a TVAR structure to model a speech signal.

It should be noted that our method differs from [15] in the sense that a TVAR *SαS process* is modeled here, i.e. both the unknown TVAR coefficients and the statistical parameters (α, σ) of a *SαS process* are estimated; whereas the main objective in [15] is to enhance a *Gaussian* TVAR signal, which is contaminated by a *SαS* process.

In its most general form, particle filters enable us to obtain the optimal Bayesian solution of the systems that can be modeled by non-Gaussian and/or nonlinear state-space equations [16, 17]. For such systems, if the signals are non-stationary, particle filters can still provide us with the optimal Bayesian solution, since the estimation is performed sequentially. However, other Bayesian techniques, such as the Markov Chain Monte Carlo (MCMC) [18], can only be used for stationary signals, since these methods have batch processing nature and discard the time information of the signals.

Motivated by these approaches, a novel method for estimating both the unknown TVAR coefficients and the statistical parameters (α, σ) of a *SαS process* is proposed in this work. The proposed method is composed of two stages, where the TVAR coefficients are estimated by a particle filter and the statistical parameters of the driving process are estimated by a Hybrid Monte Carlo method. These two stages are used iteratively starting from random values and good convergence results are obtained.

The rest of the paper is organized as follows: First, the problem is stated formally in Section II with background information on α-stable processes. Then, particle filters are introduced in Section III. In Section IV, the proposed method is presented which is followed by the experiments in Section V. Finally, results are discussed in Section VI and the conclusions are drawn in Section VII.

II. TVAR ALPHA STABLE PROCESSES

A. Alpha Stable Processes

It is well known that, if we add a large number of random variables of different distributions, the summation variable tends to be more Gaussian distributed as the number of terms goes to infinity. This is known as the Central Limit Theorem (CLT). Moreover, it is necessary that each added random variable is of finite variance. Otherwise, CLT becomes insufficient and Generalized Central Limit Theorem should be used [3]. In this case, the limiting distribution is an α-stable distribution. α-stable distributions are defined in terms of their characteristic functions, since their probability density functions (pdf) cannot be obtained analytically, except for some limited cases (α=2, β=0 Gaussian; α=1, β=0 Cauchy; α=0.5, β=-1 Pearson) [3, p. 14].The characteristic function of α-stable distributions is given as follows:

$$\varphi(t) = \exp\left\{ j\delta t - \sigma |t|^{\alpha} \left[1 + j\beta \, sign(t)\omega(t,\alpha) \right] \right\} \tag{1a}$$

Here, the parameters are defined within the following intervals: $-\infty < \delta < \infty$, $\sigma > 0$, $0 < \alpha \le 2$, $-1 \le \beta \le 1$.

$$\omega(t,\alpha) = \begin{cases} \tan\dfrac{\alpha\pi}{2} & if \quad \alpha \neq 1 \\ \dfrac{2}{\pi}\log|t| & if \quad \alpha = 1 \end{cases} \quad and \quad sign(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t = 0 \\ -1 & if \quad t < 0 \end{cases} \tag{1b}$$

As shown above, an α-stable distribution is defined by four parameters and will be represented by S(α,β,σ,δ), from now on. Among these, α and β are known as the shape parameters and they determine the thickness of the tails and the symmetry of the distribution, respectively. For example, in our work, *SαS* are used. As α gets smaller, the distributions become more impulsive. δ and σ are known as the measures of the location and the dispersion around it, respectively.

B. TVAR Alpha Stable Processes

As mentioned before, the main contribution of this work is the estimation of both the *time varying AR* coefficients and the distribution parameters of a *SαS process*, which is given in the following form:

$$y(t) = \sum_{k=1}^{K} x_k(t)\, y(t-k) + v(t) \tag{2}$$

where, y(t), $x_k(t)$ are known as the observation, autoregressive parameters and v(t) is the driving process, S(α,0,σ,0), respectively. In this work, the objective is to estimate the TVAR coefficients, $x_k(t)$, which depend on time index *t* and the statistical parameters of the *SαS* process, i.e. α and σ. Here, location parameter is taken to be zero and the shape (α) and dispersion (σ) parameters are estimated beside the TVAR coefficients. Since the α-stable processes are symmetric, β is also equal to zero. So, the characteristic function turns into the following form:

$$\varphi(t) = \exp\left\{-\sigma|t|^{\alpha}\right\} \tag{3}$$

III. PARTICLE FILTERS

Particle filters are used in order to update sequentially *a priori* knowledge about some predetermined state variables by using the observation data. In general, these state variables are the hidden variables in a non-Gaussian and nonlinear state-space modeling system. Such a system can be given by the following equations:

$$\begin{aligned} \mathbf{x}_t &= f_t\left(\mathbf{x}_{t-1}, \mathbf{v}_t\right) \\ \mathbf{y}_t &= h_t\left(\mathbf{x}_t, \mathbf{n}_t\right) \end{aligned} \tag{4}$$

where $\mathbf{x}_t$ and $\mathbf{y}_t$ represent the hidden state and the observation vectors at current time *t*, respectively. Here, the process and observation noises are denoted by $\mathbf{v}_t$ and $\mathbf{n}_t$, respectively. f_t and h_t are known as the process and observation functions and in their most general case, they are nonlinear. Also, the noise processes in (4) are modeled to be non-Gaussian. Here, the objective is to obtain sequentially the *a posteriori* distribution of the state variables obtained via the observation data gathered up to that time, i.e. $p\left(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}\right)$. If both the process and the observation noises are Gaussianly distributed and the corresponding functions f_t and h_t are linear, then the desired *a posteriori* distribution is also Gaussian and sequentially estimating the mean and variance is sufficient instead of the whole pdf. In this situation, the optimal solution can be obtained by the Kalman filter [19]. For this condition, (4) is expressed as follows:

$$\begin{aligned} \mathbf{x}_t &= \mathbf{F}_t\mathbf{x}_{t-1} + \mathbf{v}_t \\ \mathbf{y}_t &= \mathbf{H}_t\mathbf{x}_t + \mathbf{n}_t \end{aligned} \tag{5}$$

where F_t and H_t are linear operators and the noise distributions are Gaussian. For both (4) and (5), the optimal Bayesian solution for the *a posteriori* pdf is given as follows [16, 17]:

$$p\left(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}\right) = \frac{p\left(\mathbf{y}_t|\mathbf{x}_t\right)p\left(\mathbf{x}_t|\mathbf{x}_{t-1}\right)}{p\left(\mathbf{y}_t|\mathbf{y}_{t-1}\right)}\, p\left(\mathbf{x}_{0:t-1}|\mathbf{y}_{1:t-1}\right) \tag{6}$$

In general non-Gaussian situations we may not always have analytical expressions for distributions. Thus, the distributions are expressed in terms of samples, to approximate them. These samples are called as the particles. The expression for the *a posteriori* pdf can be given in terms of particles as follows:

$$p\left(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}\right) \approx \sum_{i=1}^{N} w_t^i \delta\left(\mathbf{x}_{0:t} - \mathbf{x}_{0:t}^i\right) \tag{7}$$

where w_t^i , $\mathbf{x}_{0:t}{}^{i}$, δ(.) denote the weight of the i^{th} particle, the i^{th} particle and the Kronecker delta operator, respectively. Then, expectations for function g(.) can be obtained by the following equation:

$$I\left(f_t\right) = \int g\left(\mathbf{x}_{0:t}\right) p\left(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}\right) d\mathbf{x}_{0:t} \tag{8}$$

where g(.) is a function depending on the estimate [16]. Here, the major problem is to draw samples from an analytically inexpressible non-Gaussian distribution and estimate the integral given by (8) using Monte Carlo integration techniques, shown as follows:

$$\hat{I}_N(f_t) = \sum_{i=1}^{N} g_t\left(\mathbf{x}_{0:t}^i\right)\tilde{w}_t^i \tag{9}$$

where $\tilde{w}_t^i$ denote the normalized weights given as:

$$\tilde{w}_t^i = \frac{w_t^i}{\sum_{i=1}^{N} w_t^i}, \quad i = 1, ..., N \tag{10}$$

The particles that take place in equations (7) and (9) are drawn by a method known as the "Importance Sampling" [16, 17] and the corresponding "Importance Weight" for each of them is denoted by w_t^i as defined as follows:

$$w_t^i \propto \frac{p\left(\mathbf{x}_{0:t}^i|\mathbf{y}_{1:t}\right)}{q\left(\mathbf{x}_{0:t}^i|\mathbf{y}_{1:t}\right)} \tag{11}$$

where q(.) function is called as the "Importance Function" and drawing samples from this pdf is easier than that of original distribution [16, 17]. However, importance sampling shown in (11), can be used in batch processing techniques and should be modified as follows for the sequential applications [16, 17]:

$$w_t^i \propto w_{t-1}^i \frac{p\left(\mathbf{y}_t|\mathbf{x}_t^i\right) p\left(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i\right)}{q\left(\mathbf{x}_t^i|\mathbf{x}_{0:t-1}^i, \mathbf{y}_{1:t}\right)} \tag{12}$$

But, as a consequence of this sequential modification, a phenomenon, known as "Degeneracy", arises as a problem and causes the importance weight of each particle, but one, to converge to zero as time evolves [16, 17]. In order to avoid the degeneracy problem, "Resampling" is performed as an additional step and by this procedure, particles with high importance weights are replicated, while the others are discarded. By doing so, we can approximate the desired pdf in time [16, 17].

It is well known that the optimal Bayesian estimation of unknown variables, which can be expressed in terms of a state-space system, is available through the use of particle filters [16, 17]. Although this method seems to be very effective in its most general form (nonlinear and/or Non-Gaussian), it is extremely tedious to obtain the optimal importance function to reach these optimal results, except for some special cases [17]. Usually, the designer selects a simpler proposal distribution in such cases to approximate the optimal one. Most of the time, the *a priori* transition density function is used as a proposal density, which can be given as follows:

$$q\left(\mathbf{x}_t^i|\mathbf{x}_{0:t-1}^i, \mathbf{y}_{1:t}\right) = p\left(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i\right) \tag{13}$$

Provided that this proposal distribution is used, the importance weight calculation (12), is reduced to the likelihood evaluation at the drawn sample value:

$$w_t^i \propto w_{t-1}^i p\left(y_t \mid \mathbf{x}_t^i, \boldsymbol{\theta}^{(t)}\right) \tag{14}$$

where $\boldsymbol{\theta}^{(t)}$ denotes the parameter vector of the observation noise, shown by *n(t)*. This specific particle filtering scheme is known as the Bootstrap particle filter [16, 17].

Although using the *a priori* transition density function simplifies the calculations, it usually performs unsatisfactorily, as a result of the resampling stage, after the importance weights are calculated by (14). Usually, the particle population is reduced to a few samples after this stage. This is known as the "Depletion of Samples" problem and arises as a consequence of neglecting the observation information in the process of sample transition [16, 17]. In literature, there are well known methods developed to mitigate this problem by increasing the sample diversity [20, 21].

If the *a priori* information about the states is not sufficient to construct a good approximation to the optimal importance function, one can also model the state transitions in such a way that the observation information is exploited not explicitly, but implicitly, by modeling the state transition noise with a Gaussian distribution, possessing a time-varying covariance matrix [13, 14].

IV. THE PROPOSED METHOD

The proposed method is composed of two successive sections, where the TVAR coefficients are estimated by particle filter and the statistical parameters α and σ of the *SαS* process are estimated by a Hybrid Monte Carlo method.

A. TVAR Estimation by Particle Filtering

An observed TVAR *SαS* process can be expressed by the following equation:

$$y_t = \mathbf{y}_{t-1}^T \mathbf{x}_t + n_t \tag{15}$$

where $\mathbf{y}_{t-1} = [y_{t-1},.....,y_{t-K}]^T$ and $\mathbf{x}_t = [x_1(t),.....,x_K(t)]^T$ vectors denote the past values of the AR process and the TVAR coefficient vector of order *K*, respectively. Here, the observation (driving) noise is modeled by a *SαS* process, as shown below:

$$n_t \sim S(\alpha, 0, \sigma, 0) \tag{16}$$

The objective of this work, is to model an observed TVAR *SαS* process y_t, i.e. to estimate the vector $\mathbf{x}_t$ and the statistical parameters (α and σ), jointly. Motivated by [13, 14], the unknown TVAR coefficients can be modeled as states and together with (15), they can be inserted into a state-space representation, as shown below:

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{v}_t \tag{17.a}$$

$$y_t = \mathbf{y}_{t-1}^T \mathbf{x}_t + n_t \tag{17.b}$$

where (17.a) and (17.b) are called as the state transition and observation equations, respectively. In (17.a), the state transition matrix is taken to be the identity matrix. According to this representation, distribution of the state transition (process) noise is used in order to draw new samples. In order to take the effect of the observation information into account during the state transition (sampling new particles) step, Bootstrap PF is used with the following state transition density:

$$q\left(\mathbf{x}_t \mid \mathbf{x}_{0:t-1}, \mathbf{y}_{1:t}\right) = p\left(\mathbf{x}_t \mid \mathbf{x}_{t-1}\right) = N\left(\mathbf{x}_{t-1}, \mathbf{\Sigma}_{v(t)}\right) \tag{18}$$

where $p\left(\mathbf{v}_t\right) \sim N\left(\mathbf{0}, \mathbf{\Sigma}_{v(t)}\right)$ denotes a Gaussian distribution with zero mean and covariance matrix of $\mathbf{\Sigma}_{v(t)}$. This covariance matrix is sequentially estimated from the past data in such a way that the current state estimate is affected more from recent data, while the effect of the previous data is reduced by the utilization of a "Forgetting Factor", namely ξ, as shown below:

$$\mathbf{\Sigma}_{\mathbf{v}(t)} = \mathbf{\Sigma}_{\mathbf{x}(t-1)} \left(\frac{1}{\xi} - 1\right) \tag{19}$$

where $\mathbf{\Sigma}_{\mathbf{x}(t-1)}$ is a diagonal matrix, whose elements are variances of the particles, corresponding to the related AR coefficient at time (t-1) and ξ is a real number between zero and one.

Given a state-transition equation, which is modeled by (17.a) with the proposal density function of (18), the importance weight of each particle can be calculated by using the likelihood function, given by (14). It is well known that the pdf of a *SαS* random variable, denoted by (16), can be estimated numerically by taking the inverse Fourier Transform of its characteristic function which is shown as follows:

$$p\left(n \middle| \sigma^{(t)}, \alpha^{(t)}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\sigma^{(t)} |t|^{\alpha^{(t)}}\right) \exp\left(jnt\right) dt \tag{20.a}$$

where $j = \sqrt{-1}$. In order to calculate the importance weight, pertaining to the i^{th} particle, (14) takes the following form as a result of the relationship of (15):

$$p\left(y_t \mid \mathbf{x}_t^i, \boldsymbol{\theta}^{(t)}\right) = p\left(y_t \mid \mathbf{x}_t^i, \alpha^{(t)}, \sigma^{(t)}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\sigma^{(t)} |t|^{\alpha^{(t)}}\right) \exp\left(j\left(y_t - \mathbf{y}_t^T \mathbf{x}_t^i\right)t\right) dt \tag{20.b}$$

Here, the parameter vector is composed of the shape and the dispersion parameters of the *SαS* process, as shown below:

$$\boldsymbol{\theta}^{(t)} = \left\{\alpha^{(t)}, \sigma^{(t)}\right\} \tag{21}$$

In this work, it is assumed that the statistical parameters α and σ are unknown constant values. Thus, in order to clarify that these parameters are not time-varying, their values at time *t* are shown by upper indices, denoting the algorithmic iteration number. As a result of these, the unknown statistical parameter vector, $\boldsymbol{\theta}^{(t)}$**,** must be estimated, in order to calculate the importance weights of the state variables ($\mathbf{x}_t$) representing the TVAR coefficients.

For this purpose, we make use of the heavy-tailed nature of the observation noise: TVAR coefficient estimates, $\hat{\mathbf{x}}(t)$, which are found by using random statistical parameters, $\boldsymbol{\theta}^{(t)}$ other than their true values, can still be enough to obtain a satisfactory estimate of the driving process, namely $\hat{n}(t)$, which is expressed as follows:

$$\hat{n}(t) = y_t - \mathbf{y}_{t-1}^T \hat{\mathbf{x}}(t) \approx y_t - \mathbf{y}_{t-1}^T \mathbf{x}(t) = n(t) \tag{22}$$

Thus, the proposed method proceeds as follows: First, random values are drawn from the priors of the statistical parameters, which are denoted by α_0 and σ_0, for the first iteration. Then, by using these values, the particle filtering algorithm, which is described above, is run on the observed data and an estimate of the TVAR coefficients is obtained, which is denoted by vector $\hat{\mathbf{x}}(t)$. Secondly, an estimated waveform for the *SαS* driving process (innovations), $\hat{n}(t)$, is obtained by (22).

After that, a Hybrid Monte Carlo method, which is explained in Section B, is applied on this innovations process estimate $\hat{n}(t)$, to find the statistical parameters $\alpha^{(t)}$ and $\sigma^{(t)}$.

Then, these new parameter values are given as inputs to the particle filter and the new TVAR coefficients are estimated. This procedure is performed iteratively, until both the TVAR coefficients and the statistical parameters converge.

B. Statistical Parameter Estimation by Hybrid Monte Carlo Method

Here, the method which is developed by Godsill and Kuruoğlu [11] in order to estimate the dispersion parameter of a heavy-tailed symmetric α-stable process is generalized in such a way that the shape parameter (α) can also be estimated by an additional Metropolis Hastings step. This method exploits the use of the "Positive Stable Law" which can be expressed as follows [4, 11] : Let X and λ be independent random variables with the following distributions:

$$x \sim S(\alpha', 0, \sigma, 0) \quad \text{and} \quad \lambda \sim S\left\{\alpha/\alpha', 1, \left(\cos\left(\frac{\pi\alpha}{2\alpha'}\right)^{\alpha'/\alpha}\right), 0\right\} \tag{23}$$

Then $\hat{n} = X\lambda^{1/\alpha'}$ is stable with distribution $\hat{n} \sim S(\alpha, 0, \sigma, 0)$. Moreover, a Gaussian distribution with $N\left(\mu, 2\sigma^2\right)$ corresponds to $S(2, 0, \sigma, \mu)$ with this notation. So, if we can generate positive stable random variables with distribution $\lambda \sim S\left\{\alpha/2, 1, \left(\cos\left(\frac{\pi}{4}\right)^{2/\alpha}\right), 0\right\}$ and multiply by $X \sim N\left(0, 2\sigma^2\right)$, then the multiplication $\hat{n}$ is distributed by $\hat{n} \sim N(0, 2\lambda\sigma^2)$. So, given λ and σ, $\hat{n}$ is Gaussianly distributed. Thus, given the values of λ and σ at a specific time instant *t*, the likelihood can be obtained at *t* by $N\left(\hat{n}_t \middle| 0, 2\lambda_t\sigma^2\right)$. Therefore, the posterior distribution of λ_t is given as follows, which is proportional to the multiplication of the likelihood and the prior:

$$p\left(\lambda_t \middle| \hat{n}_t, \sigma, \alpha\right) \propto N\left(\hat{n}_t \middle| 0, 2\lambda_t\sigma^2\right) S\left\{\alpha/2, 1, \left(\cos\left(\frac{\pi\alpha}{4}\right)^{2/\alpha}\right), 0\right\} \tag{24}$$

The reason of obtaining such a representation will soon be obvious. Let us think that we have gathered a batch of M α-stable random variables $\hat{n}$, i.e. $\hat{\mathbf{n}} = \left[\hat{n}_1, \hat{n}_2, ..., \hat{n}_M\right]$. Under these circumstances, the likelihood of the parameters of this distribution cannot be written analytically. However, if we can find N λ_t positive stable random variables for each $\hat{n}_t$, for $t = 1, 2, ..., M$, we can express the likelihood as follows, given the value of σ:

$$p\left(\hat{\mathbf{n}}|\boldsymbol{\lambda},\sigma,\alpha\right) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{M}\frac{\hat{n}_t^2}{\lambda_t}\right) \tag{25}$$

where $\boldsymbol{\lambda} = \left[\lambda_1, \lambda_2, ..., \lambda_M\right]^T$. Here, Gibbs sampling is utilized in order to obtain the posterior pdf's of the parameters, namely $\alpha, \sigma, \boldsymbol{\lambda}$. The following priors are used:

$$p(\sigma^2) = IG(\eta, \kappa),\ p(\alpha) = U(0, 2) \tag{26}$$

$$p(\lambda_t) = S\left\{\alpha/2, 1, \left(\cos\left(\frac{\pi\alpha}{4}\right)^{2/\alpha}\right), 0\right\} \tag{27}$$

where $IG(\eta, \kappa)$ denotes an Inverted Gamma distribution with parameters η and κ, whereas U(a,b) denotes a Uniform distribution between a and b.
By using these priors, the conditional posterior of σ^2 is obtained as follows [11]:

$$\begin{aligned} p(\sigma^2 \,|\alpha, \boldsymbol{\lambda}, \hat{\mathbf{n}}) &= IG(\eta', \kappa') \\ \eta' = \eta + M/2, \qquad \kappa' &= \kappa + \frac{1}{2}\sum_{t=1}^{M}\frac{\hat{n}_t^2}{\lambda_t} \end{aligned} \tag{28}$$

Conditional posterior of $\boldsymbol{\lambda}$ is expressed as the multiplication of conditional posterior of each λ_t:

$$p\left(\boldsymbol{\lambda}|\alpha,\sigma,\hat{\mathbf{n}}\right) = \prod_{t=1}^{N} p\left(\lambda_t|\alpha,\sigma,\hat{n}_t\right) \tag{29}$$

where $p\left(\lambda_t|\alpha,\sigma,\hat{n}_t\right)$ is given by (24). This conditional posterior can be obtained by using the Rejection Sampling scheme [22] as follows:
Target distribution is shown by $p\left(\lambda_t|\alpha,\sigma,\hat{n}_t\right)$. This can be expressed by (24), where the effect of λ_t is evaluated at the likelihood, i.e. $N\left(\hat{n}_t\middle|0, 2\lambda_t\sigma^2\right)$. So, instead of the posterior, likelihood can be taken as the target distribution. This likelihood is bounded from above [11]:

$$N\left(\hat{n}_t|0, 2\lambda_t\sigma^2\right) \le \frac{1}{\sqrt{2\pi\hat{n}_t^2}}\exp(-1/2) \tag{30}$$

Thus, the steps of the rejection sampling can be given as follows [11]:

1. $\lambda_t \sim S\left\{\alpha/2, 1, \left(\cos\left(\frac{\pi\alpha}{4}\right)^{2/\alpha}\right), 0\right\}$

2. $u \sim U\left(0, \frac{1}{\sqrt{2\pi\hat{n}_t^2}} \exp(-1/2)\right)$

3. If $u > N\left(\hat{n}_t | 0, 2\lambda_t\sigma^2\right)$ go to 1.

Finally, the conditional posterior distribution of α is obtained by using Metropolis Hastings algorithm. For this purpose, numerical likelihood functions are evaluated as in (20) for each data sample and then multiplied by the prior, as shown below:

$$p(\alpha \mid \boldsymbol{\lambda}, \sigma, \hat{\mathbf{n}}) \propto \prod_{t=1}^{N} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\sigma|\tau|^{\alpha}\right) \exp\left(j\hat{n}_t\tau\right) d\tau \right\} U(\alpha \mid 0, 2) \tag{31}$$

which corresponds to the multiplication of the likelihood and the prior. In [31], U(α|0,2) denotes the evaluation of α in the uniform distribution U(0,2). Here, Random Walk Metropolis Hastings (MH) [22, 23] is utilized, where the proposal density is chosen to be a Gaussian. The steps of this MH are given as follows:

1. Draw a new sample: $\alpha' = \alpha + N(0, \varphi^2)$, where φ^2 denotes the variance of the random jumps [22].

2. Calculate the acceptance ratio:

$$\rho = \min\left(1, \frac{p(\alpha' \mid \boldsymbol{\lambda}, \sigma, \hat{\mathbf{n}})}{p(\alpha \mid \boldsymbol{\lambda}, \sigma, \hat{\mathbf{n}})}\right)$$

3. Accept this new α' with probability ρ.

After obtaining the necessary conditional posteriors, Gibbs sampling is performed by iterating through the following samplings:

1. $\boldsymbol{\lambda}^{i+1} \sim p\left(\boldsymbol{\lambda} \middle| \hat{\mathbf{n}}, \sigma^i, \alpha^i\right)$ (32.1)

2. $\sigma^{2(i+1)} \sim p\left(\sigma^2 \middle| \alpha^i, \boldsymbol{\lambda}^{i+1}, \hat{\mathbf{n}}\right)$ (32.2)

3. $\alpha^{i+1} \sim p\left(\alpha | \boldsymbol{\lambda}^{i+1}, \sigma^{i+1}, \hat{\mathbf{n}}\right)$ (32.3)

After iterating through these, take the average of each parameter as the point estimate, which is calculated after the burn-in period [22]:

$$\boldsymbol{\lambda} = \frac{1}{L}\sum_{i=1}^{L} \boldsymbol{\lambda}^i, \sigma = \frac{1}{L}\sum_{i=1}^{L} \sigma^i, \alpha = \frac{1}{L}\sum_{i=1}^{L} \alpha^i \tag{32.4}$$

where L denotes the number of iterations, after the values pertaining to the burn-in period are omitted.

V. EXPERIMENTS

A. *The Proposed Method*

In this section, performance of the proposed method is shown by computer simulations, where five different experiments are performed with the specifications outlined in Table I.

Synthetic TVAR processes are generated by passing each of the driving processes, whose distributions are given on the second column of Table I, through a first order all-pole filter. It should be noted that, (19) takes the following form in case of a first order AR model:

$$\sigma_{v(t)}^2 = \sigma_{x(t-1)}^2 \left(\frac{1}{\xi} - 1\right) \tag{33}$$

The third, fourth and the fifth columns of Table I show the forgetting factors, total number of particles and the time variation of the AR coefficients, respectively. The value of the pole (AR coefficient) changes from 0.99 to 0.95 at t = 500 and varies sinusoidally between 0 and 1, for "jump" and "sine" type of indications, respectively. The estimates of both the TVAR coefficients and the statistical parameters are shown in Figs. 1 through 5, with their corresponding experiment number. In each figure, the first and the fourth iterations of the proposed algorithm are illustrated for the estimated coefficients and the parameters.

B. Artificial State Transition Modeling for the Static Parameters

It is well known that the particle filter is used for modeling dynamic systems, where the state variables do change in time [16, 17]. In our work, the objective is to estimate both the TVAR coefficients and the static parameters (α and σ), jointly. In order to do so, one can also propose to augment the state vector (TVAR coefficient) with the static parameters and model them with an artificial transition equation. However, it is also well known that trying to fit an artificial state transition model to a static parameter within a particle filtering framework, provides unsatisfactory results [24]. In order to show this situation, the shape parameter is also modeled by a state transition equation as shown below. Here, to make things simpler, dispersion parameter is assumed to be known.

$$\begin{aligned} x_t &= x_{t-1} + v_t \\ \alpha_t &= \alpha_{t-1} + \eta_t \\ y_t &= \mathbf{y}_{t-1}^T x_t + n_t \end{aligned} \tag{33}$$

In this experiment, the variance of the state process v_t is chosen as outlined in the previous section, while the variance of the second state process noise, namely var(η_t) is taken to be a small constant around $4x10^{-4}$. In this scenario, n_t is synthetically produced by n_t~S(1.2,0,1,0). Again a first order AR process is simulated and time variation of the AR coefficient is taken to be as a jump at t = 500. The estimations of the TVAR and the shape parameter are shown in Fig. 6.

C. Heavy Tailed Innovations Process Estimation

As outlined in Section IV, in our method, estimates of the unknown TVAR coefficients are used to obtain an approximation to the innovations process, by (22). This is performed during each iteration of the algorithm and provides satisfactory estimation of the innovations process as iterations evolve in time. In this experiment, the robustness of the AR coefficient estimation for heavy tailed innovations is visualized. For this purpose, the likelihood functions of the AR coefficient, which are used to calculate the importance weights by (20.b) are plotted at each time instant *t*, given the observed data values, namely y_t and y_{t-1}. In this experiment, AR coefficient is taken to be constant at 0.95 and the behavior of the aforementioned likelihood functions for different *SαS* driving process (innovations) are visualized for three different processes in Figs. 7 through 9, respectively.

In addition to these, in order to illustrate that one can obtain satisfactory innovations process estimate from (22), the innovations process estimate, pertaining to the first iteration of experiment 1 (Fig. 1 (b1).) is shown in Fig. 10.

VI. DISCUSSION

In this paper, a novel method is proposed to estimate both the TVAR coefficients and the statistical parameters of *SαS* processes. For this purpose, iterative utilization of a particle filter and Hybrid Monte Carlo stages are performed and successful results are obtained, as shown in Figs. 1 through 5. In this method, particle filter is used to estimate the TVAR coefficients, while a Hybrid Monte Carlo method is utilized for the estimation of the statistical parameters.

Here, it is also shown that estimating time-varying states can be successfully performed, whereas large variations arise in the estimates of the static parameters when the latter ones are also interpreted as time-varying states. This is shown in Fig. 6 and coincides with the discussion in [24], where it is stated that modeling the static parameters in a dynamic way ("artificial evolution") causes an "information loss". To avoid this phenomenon "kernel smoothing of parameters" is proposed in [24].

In our case, to estimate the static driving noise parameters, the heavy tailed nature of the signal is exploited and the static parameters can also be estimated successfully, as an alternative to those performed in [24]. Here, we make use of a nice behavior that is observed in *SαS* processes. This property can be understood by examining Fig. 1, where it is seen that the quality of the TVAR estimate increases suddenly in Fig. 1b. when a high-valued outlier is observed in Fig. 1a. This sounds very logical, since an outlier from the tails of a distribution brings us more information compared to that is brought from a sample around the mean value. Thus, as the TVAR process becomes more heavy tailed, the probability that an outlier is observed increases, resulting in

better estimates regarding to the TVAR coefficients. Even if the value of the observed TVAR process decreases, the quality of the estimated TVAR coefficients is not affected considerably.

This robustness of the TVAR estimates is also visualized in Figs. 7 through 9, where the likelihood functions for three different *SαS* processes are shown, both in 3 dimensional mesh and their corresponding 2 dimensional contour graphs. For the sake of a better understanding, a first order AR process having a *time-invariant* AR coefficient at 0.95 is used. For comparison, three different *SαS* processes are used to generate the AR processes. In Figs. 7, 8 and 9, it is seen that, as the process becomes more heavy tailed, the number of outlier data increases and the likelihood functions corresponding to the AR coefficients become well-shaped, i.e. they become sharper, which leads to a better selection between the samples drawn from the support region of the AR coeficients ([-1,1]), during the importance sampling. Moreover, the nature of the state-transition equation (17.a) allows the states, which are pulled out to a suitable region by the help of an outlier, remain for a considerable time at that value and does not allow it to jump to random values, thanks to the forgetting factor in (19). Additionally, once the effect of the outlier is forgotten, it is also harder for the state to jump to another value, as the driving process becomes more heavy-tailed. Here, the possibility of the state jump from a well-defined value (AR estimates which are found by outlier data values) increases when data values approach to zero and this phenomenon can occur more easily as we go from Fig. 7 to 9. That is why, the quality of the AR coefficient estimates decreases as the process becomes less heavy-tailed. However, this behavior can be avoided by increasing the number of particles that are used.

These properties of *SαS* processes allow us to obtain satisfactory estimates of the innovations process, which is given by (22) and lets us the ability to use MCMC methods to find the statistics of the process. The error that is done in the estimation of the innovations is very small when compared to the values that the process can take. This is illustrated in Fig. 10, where the error at the end of the fourth algorithmic iteration[1] is visualized.

VII. CONCLUSIONS

In this work, a novel method is proposed to estimate both the TVAR coefficients and the statistical parameters of a *SαS* process, jointly. This is a generalization of the case, where the statistical parameters of the *SαS* process are assumed to be known [25].

The proposed algorithm is composed of two interacting, iterative stages, where, TVAR coefficients are estimated by particle filter, while a Hybrid Monte Carlo method is utilized for the estimation of the statistical parameters of the *SαS* process. As time evolves, the estimation quality of the TVAR coefficients and the statistical parameters increase and successful convergence results are observed. It is well known that dynamic modeling can be performed very well by the utilization of the particle filter, whereas this method provides very inconsistent estimates when the static parameters are also treated within the particle filtering framework [24]. Here, this situation is also shown with an additional simulation, where the static distribution parameter of the *SαS* process is found to be very oscillatory, while the TVAR coefficient is found successfully. On the other hand, our method is a very promising solution to mitigate such problems, when the process has a *SαS* distribution.

It is also shown that, the AR estimation is very robust to the statistical parameter values of the *SαS* process, which is a similar conclusion stated for the Maximum Likelihood estimation of *time-invariant* AR α-stable processes [26].

In conclusion, the successful performance of the developed method serves as a promising contribution in the modeling of impulsive signals, which are frequently seen in many areas, such as teletraffic in computer communications, radar and sonar applications and mobile communications.

Moreover, this work presents various aspects of the behavior of particle filtering algorithms, where α-stable processes are involved, and provides material for the examination of such cases in the future.

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[1]One algorithmic iteration is composed of one particle filtering to find a TVAR estimate sequence plus the MCMC iterations to obtain the statistics to be given as input for the next particle filtering

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TABLE I
EXPERIMENT CRITERIA

Experiment Number	Distribution	ξ	N	Change Type
1	$n_t \sim S(1.1, 0, 1.5, 0)$	0.9	100	Jump
2	$n_t \sim S(1.5, 0, 1.5, 0)$	0.96	100	Jump
3	$n_t \sim S(1.9, 0, 1.5, 0)$	0.96	500	Jump
4	$n_t \sim S(1.1, 0, 1.5, 0)$	0.9	100	Sine
5	$n_t \sim S(1.9, 0, 1.5, 0)$	0.9	100	Sine

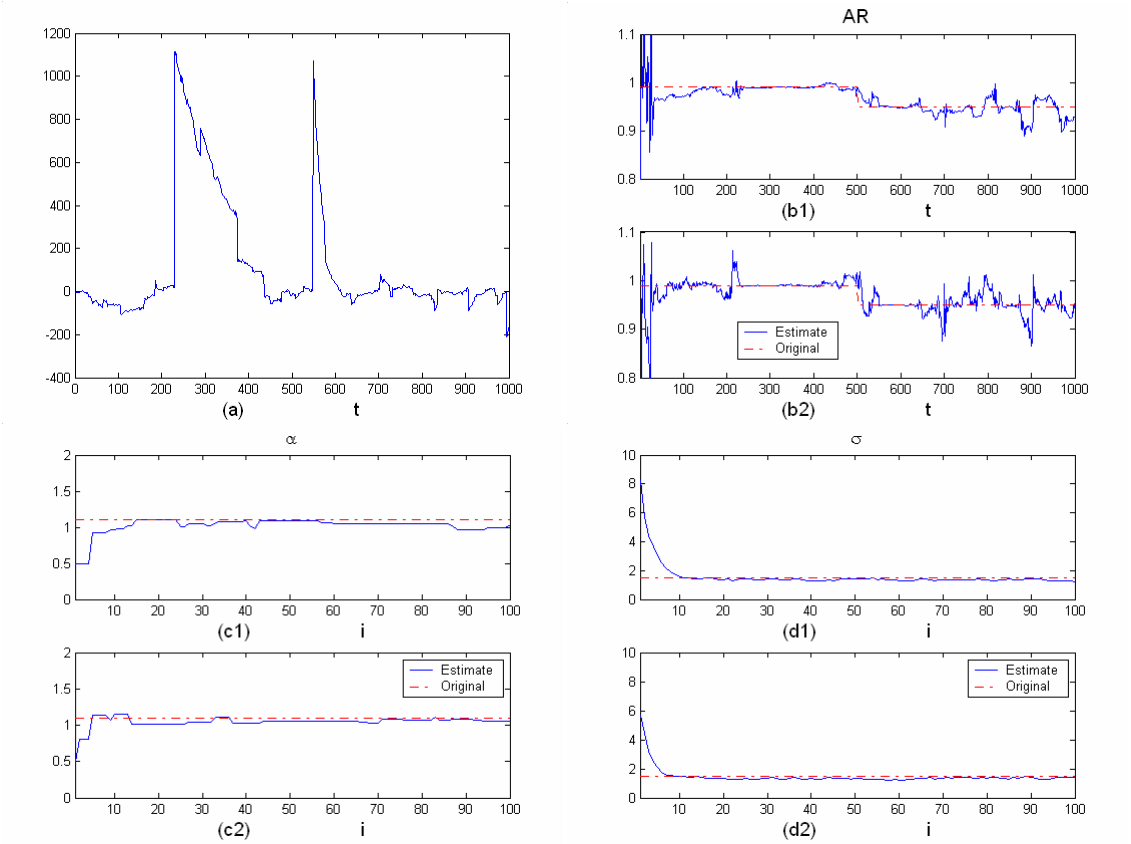

Fig.1. Experiment 1: Estimation of piecewise constant AR coefficient and the statistical parameters of $S\alpha S$ process with driving process distribution of $n_t \sim S(1.1, 0, 1.5, 0)$, where t denotes time and i denotes MCMC iterations

a) TVAR process, b1) TVAR coefficient estimate at the end of algorithmic iteration #1, b2) TVAR coefficient estimate at the end of algorithmic iteration #4, c1) Shape parameter estimate at the end of algorithmic iteration #1, c2) Shape parameter estimate at the end of algorithmic iteration #4, d1) Dispersion parameter estimate at the end of algorithmic iteration #1, d2) Dispersion parameter estimate at the end of algorithmic iteration #4

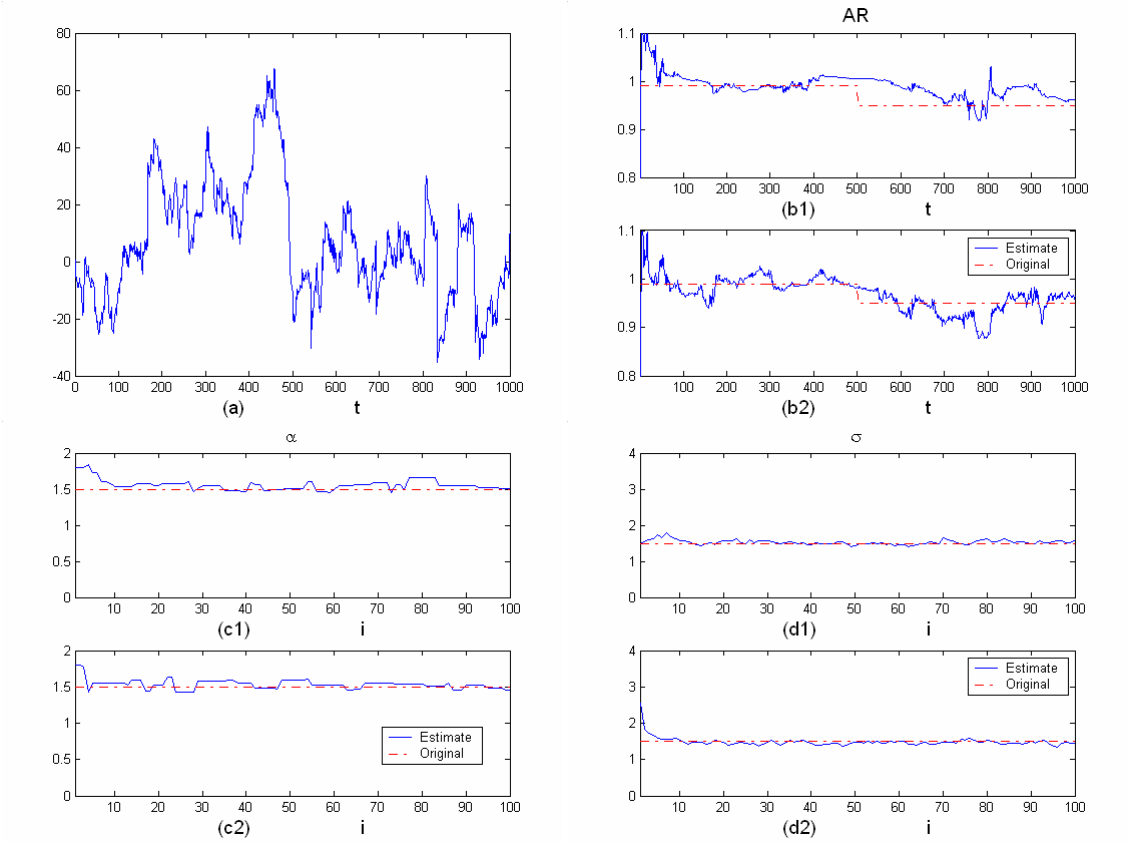

Fig.2. Experiment 2: Estimation of piecewise constant AR coefficient and the statistical parameters of $S\alpha S$ process with driving process distribution of $n_t \sim S(1.5, 0, 1.5, 0)$, where t denotes time and i denotes MCMC iterations

a) TVAR process, b1) TVAR coefficient estimate at the end of algorithmic iteration #1, b2) TVAR coefficient estimate at the end of algorithmic iteration #4, c1) Shape parameter estimate at the end of algorithmic iteration #1, c2) Shape parameter estimate at the end of algorithmic iteration #4, d1) Dispersion parameter estimate at the end of algorithmic iteration #1, d2) Dispersion parameter estimate at the end of algorithmic iteration #4

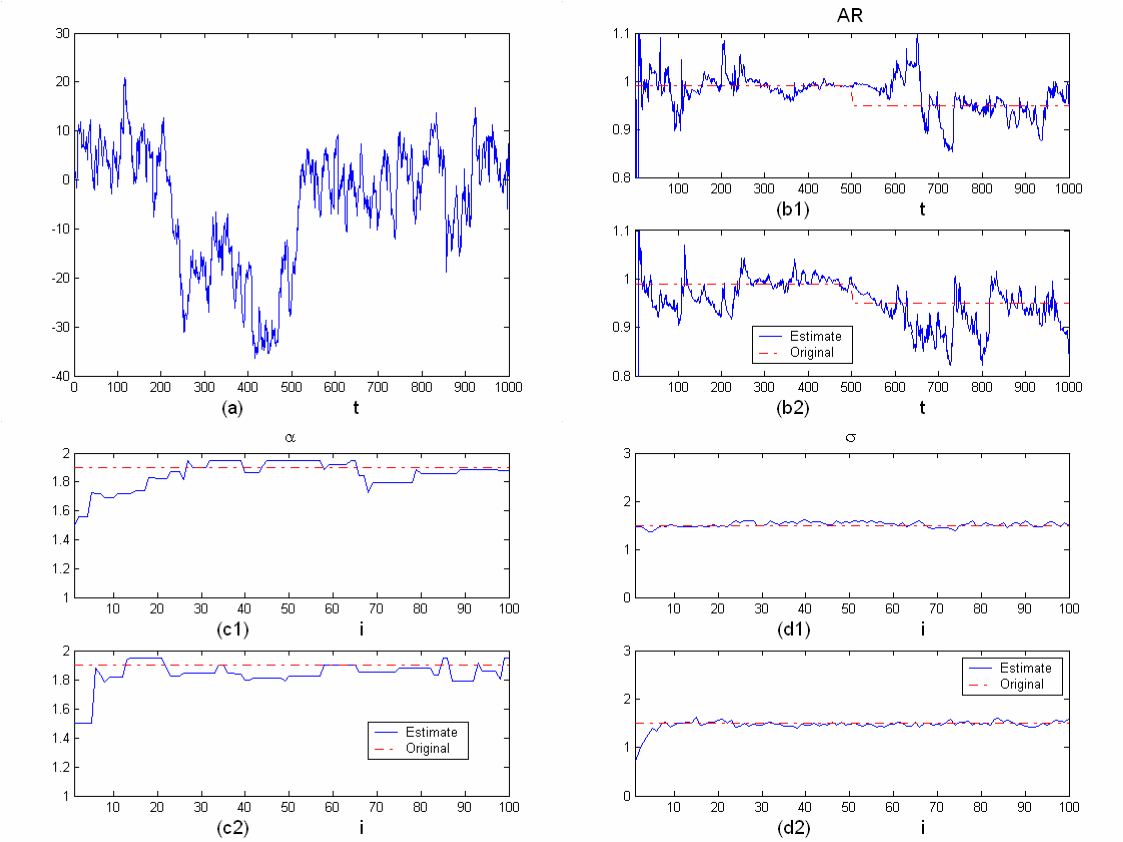

Fig.3. Experiment 3: Estimation of piecewise constant AR coefficient and the statistical parameters of $S\alpha S$ process with driving process distribution of $n_t \sim S(1.9, 0, 1.5, 0)$, where t denotes time and i denotes MCMC iterations

a) TVAR process, b1) TVAR coefficient estimate at the end of algorithmic iteration #1, b2) TVAR coefficient estimate at the end of algorithmic iteration #4, c1) Shape parameter estimate at the end of algorithmic iteration #1, c2) Shape parameter estimate at the end of algorithmic iteration #4, d1) Dispersion parameter estimate at the end of algorithmic iteration #1, d2) Dispersion parameter estimate at the end of algorithmic iteration #4

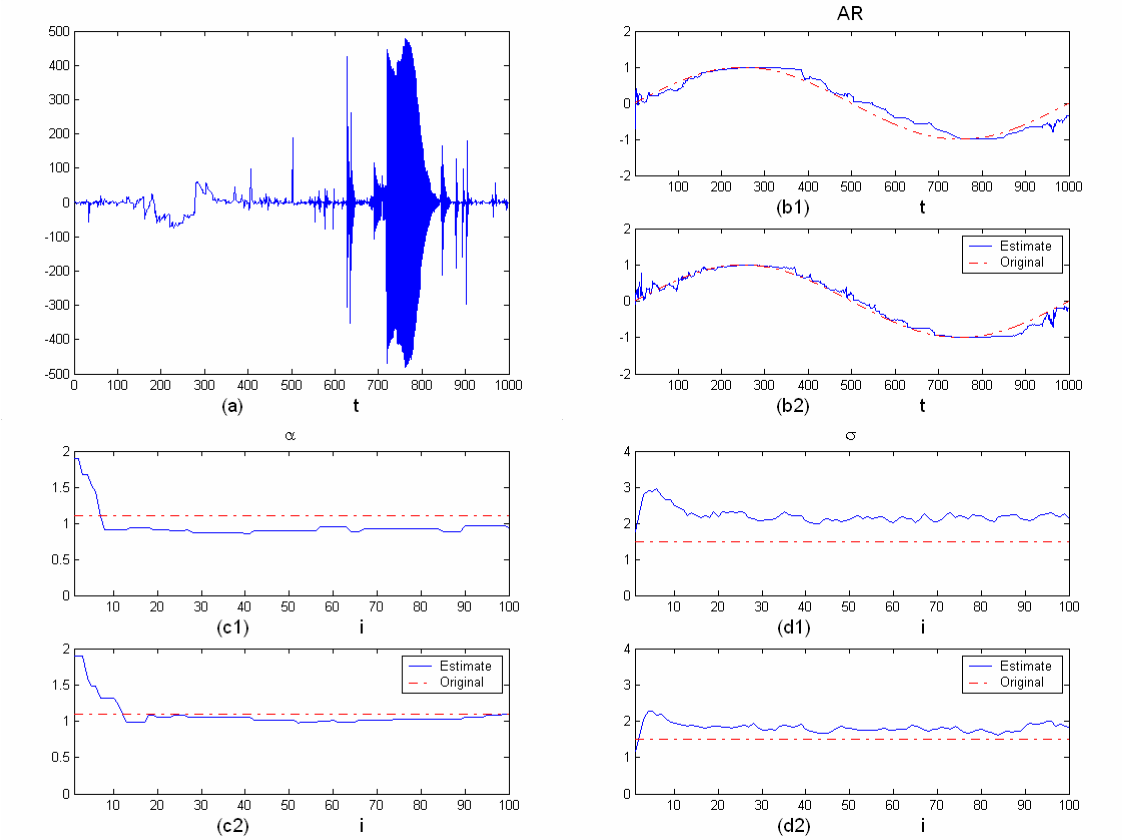

Fig.4. Experiment 4: Estimation of sinusoidal AR coefficient and the statistical parameters of $S\alpha S$ process with driving process distribution of $n_t \sim S(1.1, 0, 1.5, 0)$, where t denotes time and i denotes MCMC iterations

a) TVAR process, b1) TVAR coefficient estimate at the end of algorithmic iteration #1, b2) TVAR coefficient estimate at the end of algorithmic iteration #4, c1) Shape parameter estimate at the end of algorithmic iteration #1, c2) Shape parameter estimate at the end of algorithmic iteration #4, d1) Dispersion parameter estimate at the end of algorithmic iteration #1, d2) Dispersion parameter estimate at the end of algorithmic iteration #4

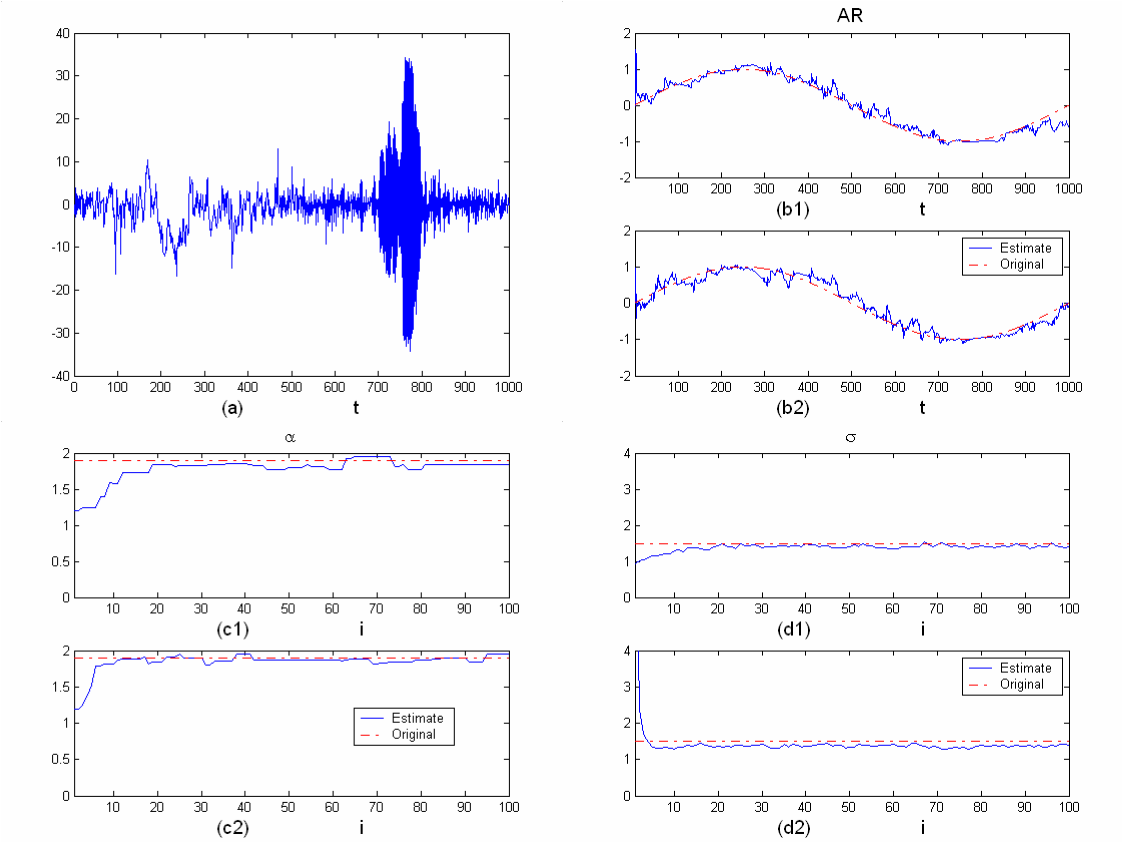

Fig.5. Experiment 5: Estimation of sinusoidal AR coefficient and the statistical parameters of $S\alpha S$ process with driving process distribution of $n_t \sim S(1.9, 0, 1.5, 0)$, where t denotes time and i denotes MCMC iterations

a) TVAR process, b1) TVAR coefficient estimate at the end of algorithmic iteration #1, b2) TVAR coefficient estimate at the end of algorithmic iteration #4, c1) Shape parameter estimate at the end of algorithmic iteration #1, c2) Shape parameter estimate at the end of algorithmic iteration #4, d1) Dispersion parameter estimate at the end of algorithmic iteration #1, d2) Dispersion parameter estimate at the end of algorithmic iteration #4

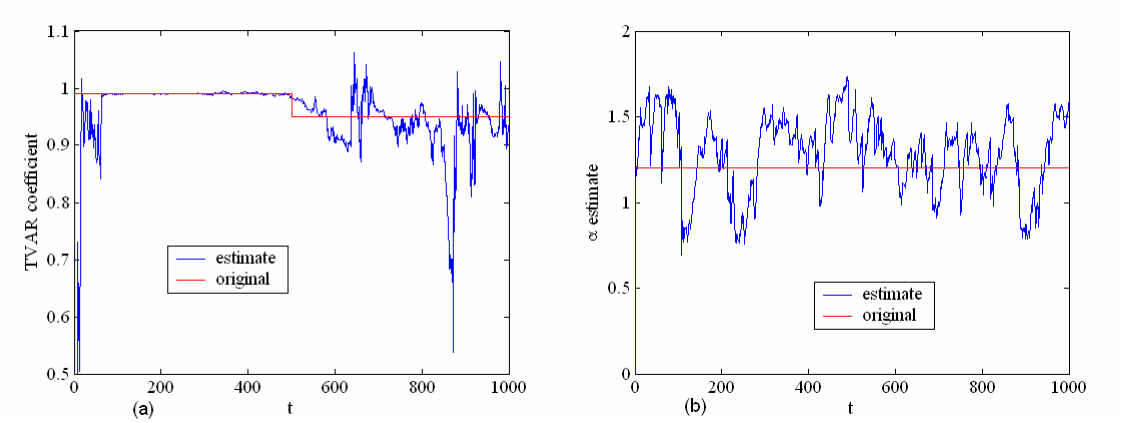

Fig. 6. Artificial State Transition Modeling for the Static Parameters, a) Estimation of piecewise constant AR coefficient and the statistical parameters of SαS process with driving process distribution of $n_t \sim S(1.2, 0, 1, 0)$ within the same particle filtering scheme where t denotes time.

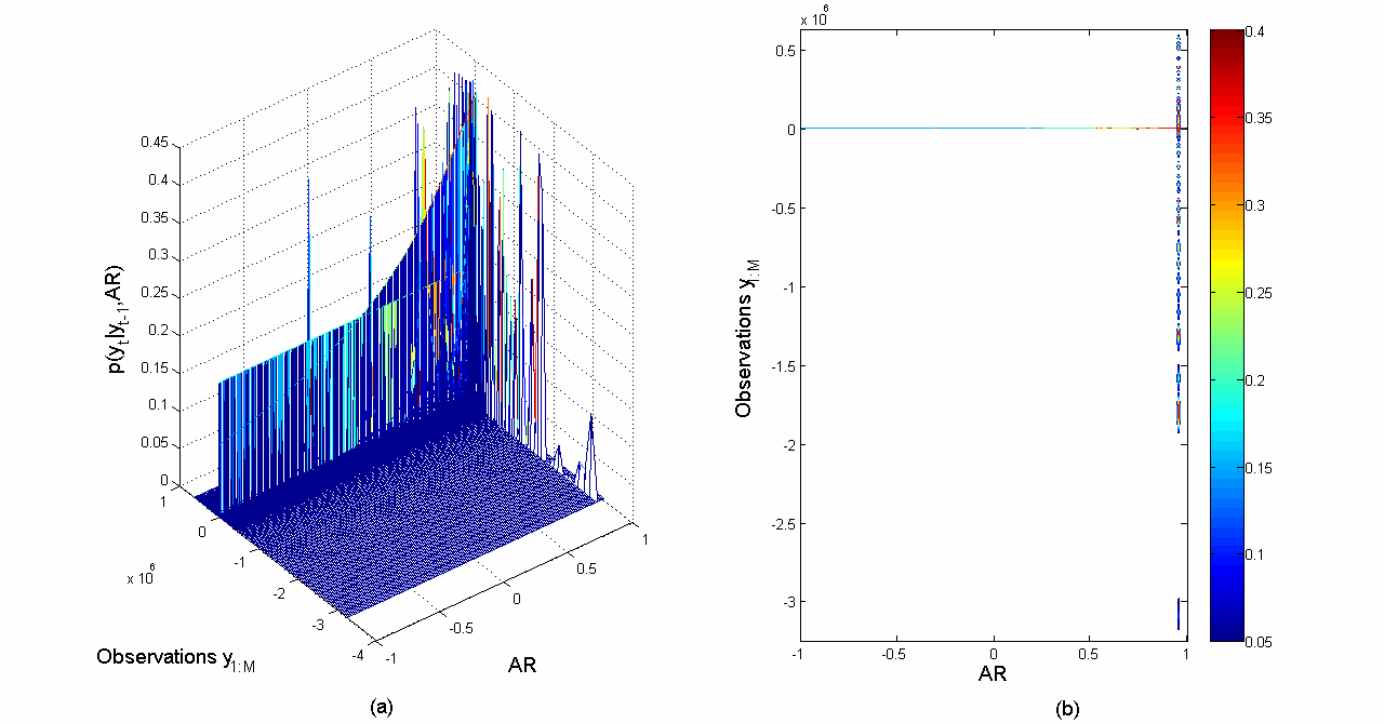

Fig. 7. Importance weights for each data $y_{1:M}$ = Likelihood functions of the AR coefficient at each time instant, i.e. t = 1:M for a synthetically generated time-invariant, first order AR process, with AR coefficient 0.95, which is driven by a *SαS* process with driving process distribution of $n_t \sim S(0.5, 0, 1.5, 0)$

a) Mesh plot, b) Contour plot

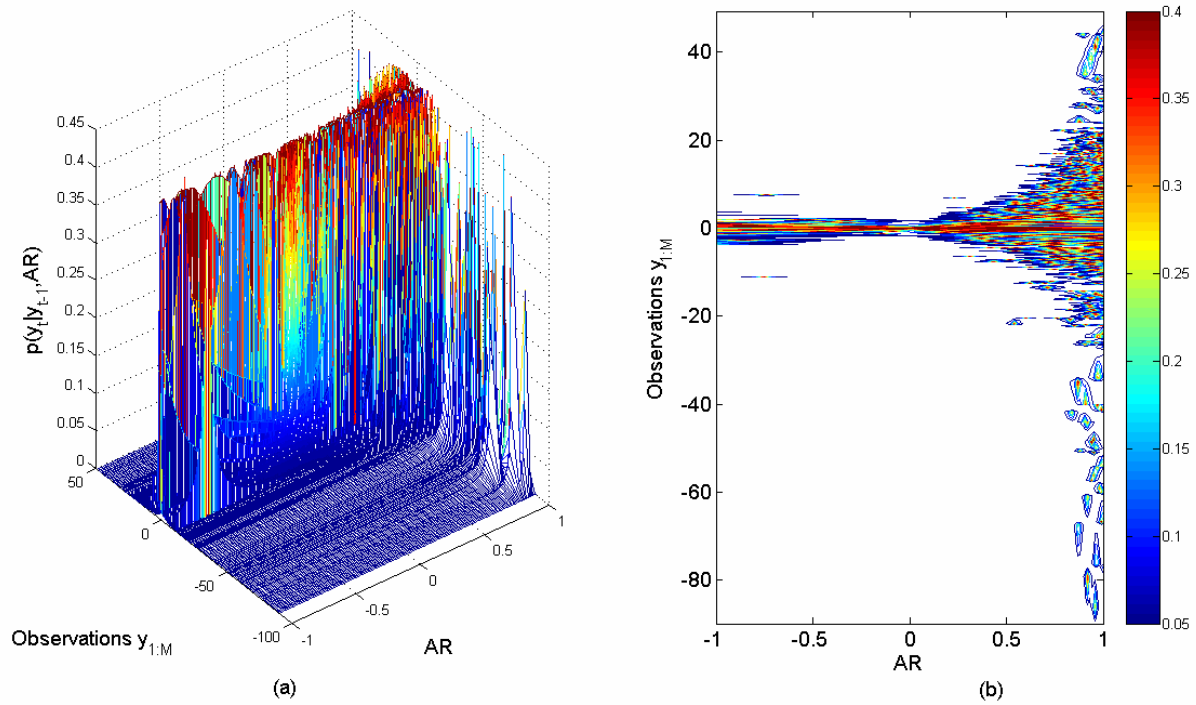

Fig. 8. Importance weights for each data $y_{1:M}$ = Likelihood functions of the AR coefficient at each time instant, i.e. t = 1:M for a synthetically generated time-invariant, first order AR process, with AR coefficient 0.95, which is driven by a $S\alpha S$ process with driving process distribution of $n_t \sim S(1.5,0,1.5,0)$

a) Mesh plot, b) Contour plot

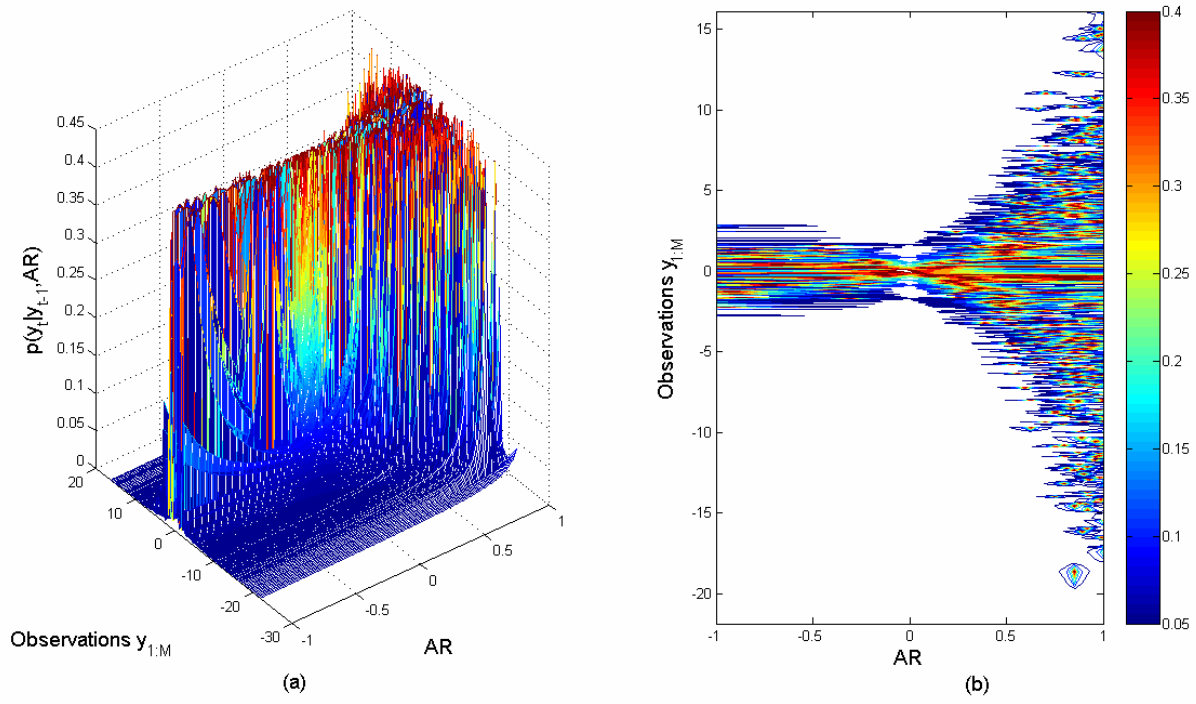

Fig. 9. Importance weights for each data $y_{1:M}$ = Likelihood functions of the AR coefficient at each time instant, i.e. t = 1:M for a synthetically generated time-invariant, first order AR process, with AR coefficient 0.95, which is driven by a $S\alpha S$ process with driving process distribution of $n_t \sim S(2,0,1.5,0)$

a) Mesh plot, b) Contour plot

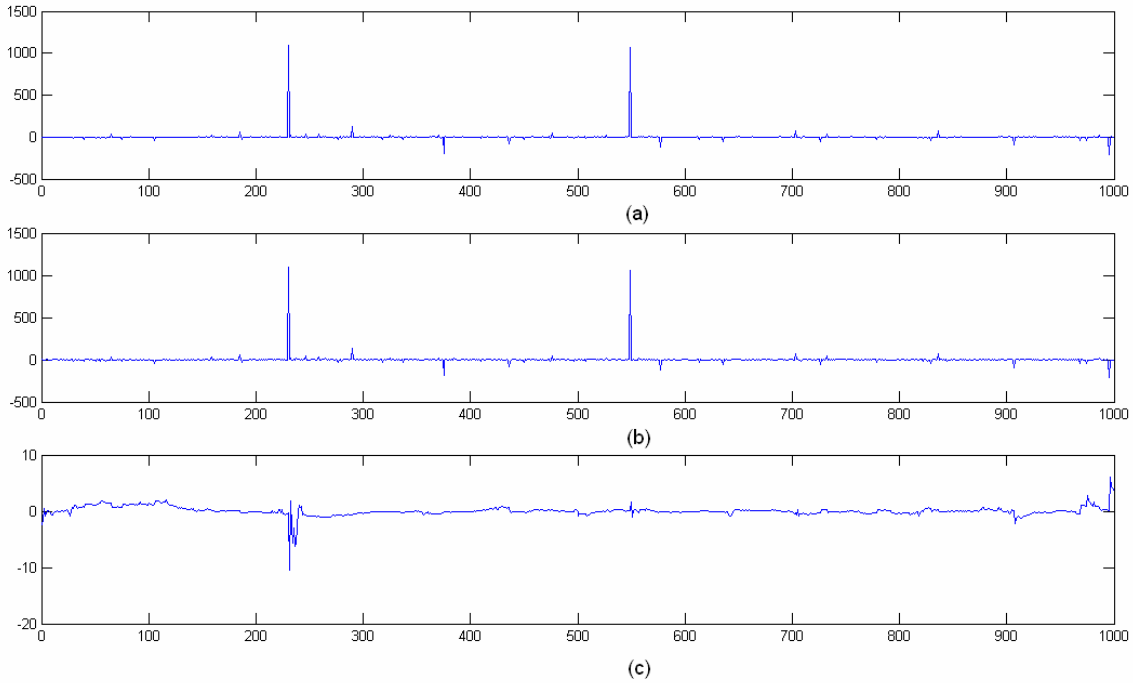

Fig. 10. a) Original driving process of Experiment 1: S(1.1,0,1.5,0) versus time, b) Its estimate at the end of the algorithmic iteration #1, c) Difference signal between (a) and (b)